HEAT TRANSMISSION THROUGH OUTSIDE WALLS

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The problem of heat transmission in multi-layer walls is analyzed. It is shown that the officially recommended method requires improvement and new design formulas are proposed.

Building materials are capillary-porous bodies containing moisture in the form of liquid, vapor, and, at sub-zero temperatures, ice. It has been shown [1, 5] that the problem of heat transmission in buildings is closely related to questions of heat and mass transfer. Therefore, to find the heat transmission through outside walls, it is necessary to solve the complete heat and mass transfer equations:

$$\frac{\partial(\gamma u_i)}{\partial\tau} = -\operatorname{div} j_i + I_i, \tag{1}$$

$$c\gamma\frac{\partial t}{\partial\tau} = \operatorname{div}(\lambda\Delta t) + r_i I_i. \tag{2}$$

Here we shall neglect heat transfer due to filtration of vapor-gas mixture. It has been shown [4] that this assumption is justified for $K_f < 0.5$. In the positive temperature range $t > 0°C$ the liquid source I_2 ($I_2 = -I_1$) is conditioned by condensation of vapor. According to Posnov's calculations, under normal conditions the mass of moist air in the capillaries is 10^{-5}% of the mass of liquid corresponding to equilibrium moisture content. Then it is accurate enough to make the specific mass content equal to the specific mass content of liquid ($u = u_2$). From Eq. (1) we obtain

$$I_1 = \operatorname{div} j_i = -\operatorname{div}\left(\varepsilon\rho D\nabla p_{10}\frac{M_1}{M}\right), \tag{3}$$

where ε is the resistance coefficient for internal diffusion of vapor, and D the diffusion coefficient of vapor in air. For a building material in the moist state, the partial pressure of vapor (p_{10}) in the capillaries is equal to the saturation pressure p_s and is a function of temperature. Then

$$I_1 = -\varepsilon\rho D\frac{M}{pM}\frac{\partial p_s}{\partial T}\nabla^2 t. \tag{4}$$

Assuming that the thermophysical characteristics depend very little on the coordinates, we may write (2) in the form

$$\frac{\partial t}{\partial\tau} = a_{eq}\frac{\partial^2 t}{\partial x^2}, \tag{5}$$

where a_{eq} is the equivalent thermal diffusivity:

$$a_{eq} = \left(\lambda + r\varepsilon D_{12}\frac{RT}{M_1}\frac{\partial p_s}{\partial T}\right)\frac{1}{c\gamma}. \tag{6}$$

It is shown in [4, 5] that (5) may be used to calculate the heat transmission through outside walls, when a_{eq}, calculated from (6), is taken as a. In the presence of a considerable moisture gradient over the thickness of the wall, it should be divided into layers, and the mean characteristics of each layer used.

We shall employ (5) to analyze heat transmission through a multi-layer wall. Heat transmission through an outside wall is usually unsteady. In the first approximation we shall assume that the temperature of the outside air (outside surface) varies according to a harmonic law.

Mathematically, the problem of an n-layer wall is described by the following system of equations:

$$\frac{\partial t_1}{\partial\tau} = a_{1eq}\frac{\partial^2 t_1}{\partial x^2},\quad 0 < x < l_1;$$

$$\cdots\cdots\cdots\cdots\cdots \tag{7}$$

$$\frac{\partial t_n}{\partial\tau} = a_{neq}\frac{\partial^2 t_n}{\partial x^2},\quad l_{n-1} < x < l_n;$$

$$t_i(\tau,\ l_i) = t_{i+1}(\tau,\ l_i);$$
$$\frac{\lambda_i}{\lambda_{i+1}}\ \frac{\partial t_i}{\partial x} = \left.\frac{\partial t_{i+1}}{\partial x}\right|_{x=l_i}\ (i = 1,\ 2,\ \ldots,\ n-1) \tag{8}$$

with appropriate boundary and initial conditions. In [6, 7] Eqs. (7) have been solved for a two-layer wall, and in [8] for the n-layer case.

In this paper we shall examine Eqs. (11) and (12) from [6], and (5), (6), and (7) from [8].

We shall write expressions for the temperatures and heat fluxes in two-layer and three-layer systems.

Two-layer wall. An outside layer of thickness l_1 is in perfect thermal contact with a second layer of considerably greater thickness ($l_2 \gg l_1$); we may assume $l_2 \to \infty$. The initial temperature is constant (or equal to zero). Starting from time $\tau = 0$, the temperature of the outside surface changes according to an harmonic law, $t(0,\ \tau) = t_m \sin \omega\tau$; $t(\infty,\ \tau) = t_c$. The solution of (7) has the form [6]:

$$\frac{t_1(x,\ \tau) - t_c}{t_m} = \exp(-\sqrt{\omega/2\,a_1}\ x)\left(\frac{\Delta_x}{\Delta_0}\right)^{1/2} \sin(\omega\tau - \sqrt{\omega/2\,a_1}\ x - \beta + \beta_x); \tag{9}$$

$$\frac{t_2(x,\ \tau) - t_c}{t_m} = \exp[-\sqrt{\omega/2\,a_1}\ l_1 - \sqrt{\omega/2\,a_2}(x - l_1)] \times$$
$$\times \frac{1-h}{\Delta_0^{1/2}}\ \sin[\omega\tau - \sqrt{\omega/2a_1}\ l_1 - \sqrt{\omega/2a_2}\ (x - l_1) - \beta]; \tag{10}$$
$$\Delta_x = 1 - 2h\exp[-2(l_1 - x)\sqrt{\omega/2a_1}]\ \cos 2(l_1 - x)\sqrt{\omega/2\,a_1} +$$
$$+ h^2\exp[-4(l_1 - x)\sqrt{\omega/2a_1}];$$
$$\Delta_0 = \Delta_x|_{x=0},$$

where

$$h = (1 - K_\varepsilon)/(1 + K_\varepsilon);\ K_\varepsilon = (\lambda_1 c_1 \gamma_1/\lambda_2 c_2 \gamma_2)^{1/2};$$
$$\beta_x = \arccos\frac{1 - h\exp[-2(l_1 - x)\sqrt{\omega/2a_1}]\ \cos 2(l_1 - x)\sqrt{\omega/2a_1}}{\Delta_x^{1/2}}; \tag{11}$$
$$\beta = \beta_x|_{x=0}.$$

For the heat fluxes we obtain

$$q_1(x,\ \tau) = \big(\sqrt{\lambda_1 c_1 \gamma_1 \omega}\ \{1 + 2h\exp[(-2l_1 + 2x)\sqrt{\omega/2a_1}]\ \cos 2(l_1 - x) \times$$
$$\times \sqrt{\omega/2a_1} + h^2\exp[(-4l_1 + 4x)\sqrt{\omega/2a_1}]\}\ \{1 - 2h\exp(-2l_1 \times$$
$$\times \sqrt{\omega/2a_1})\ \cos 2l_1\sqrt{\omega/2a_1} + h^2\exp(-4l_1\sqrt{\omega/2a_1})\}^{-1}\big)^{1/2} \times$$
$$\times \exp(-\sqrt{\omega/2a_1}\ x)\ \sin(\omega\tau - \sqrt{\omega/2a_1}\ x - \beta - \beta_x + \pi/4); \tag{12}$$
$$q_2(x,\ \tau) = \sqrt{\lambda_2 c_2 \gamma_2 \omega}\ \exp[-\sqrt{\omega/2a_1}\ l_1 - \sqrt{\omega/2a_2}\ (x - l_1)](1 - h) \times$$
$$\times [1 - 2h\exp(-2l_1\sqrt{\omega/2a_1})\ \cos 2l_1\sqrt{\omega/2a_1} + h^2\exp(-4l_1 \times$$
$$\times \sqrt{\omega/2a_1})]^{-1/2}\ \sin(\omega\tau - \sqrt{\omega/2a_1}\ l_1 - \sqrt{\omega/2a_2}\ (x - l_1) - \beta + \pi/4). \tag{13}$$

Three-layer wall. The temperature at the outside surface varies according to an harmonic law $t_1(0,\tau) = t_0 + t_m \cos \omega\tau$, and heat transfer takes place at the inside surface according to Newton's law, $\lambda_3(dt_3/dx) = -\alpha(t_3 - t_c)|_{x=l_3}$. The temperature of the inside air remains constant. The initial temperature is taken as the temperature established in the wall when the temperature of the outside surface $t_s = t_0' + t_m$, and heat transfer at the inside surface is governed by Newton's law. The solution of (7) has the form [8]:

$$t_1(x,\ \tau) = t_{1cx} + t_m A_1(x,\ \omega)\cos(\omega\tau - \gamma + \beta_{1x}); \tag{14}$$
$$t_2(x,\ \tau) = t_{2cx} + 2K_1 t_m A_2(x,\ \omega)\cos(\omega\tau - \gamma + \beta_{2x}); \tag{15}$$
$$t_3(x,\ \tau) = t_{3cx} + 4K_1 K_2 t_m A_3(x,\ \omega)\cos(\omega\tau - \gamma + \beta_{3x}). \tag{16}$$

For the heat fluxes we have

$$q_1(x,\ \tau) = Q_{1c} - t_m A_1(x,\ \omega)\cos(\omega\tau - \gamma + \beta_{1x} + \beta_x')\sqrt{\lambda_1 c_1 \gamma_1 \omega \frac{\Delta_x'}{\Delta_x}}; \tag{17}$$

$$q_2(x,\ \tau) = Q_{2c} - 2K_1 t_m A_2(x,\ \omega)\cos(\omega\tau - \gamma + \beta_{2x} + \beta_{2x}') \times \times\sqrt{\lambda_2 c_2 \gamma_2 \omega \frac{\Delta_{2x}'}{\Delta_{2x}}}; \tag{18}$$

$$q_3(x,\ \tau) = Q_{3c} - 4K_1 K_2 t_m A_3(x,\ \omega)\cos(\omega\tau - \gamma + \beta_{3x} + \beta_{3x}') \times \times\sqrt{\lambda_3 c_3 \gamma_3 \omega \frac{\Delta_{3x}'}{\Delta_{3x}}}, \tag{19}$$

where

$$A_1(x,\ \omega) = (\Delta_x/\Delta_0)^{1/2};\ A_2(x,\ \omega) = (\Delta_{2x}/\Delta_0)^{1/2};\ A_3(x,\ \omega) = (\Delta_{3x}/\Delta_0)^{1/2};$$

$$t_{1cx} = t_0' + \frac{R_x}{R_0}(t_c - t_0');\ t_{2cx} = t_0' + \frac{R_1 + R_{2x}}{R_0}(t_c - t_0'); \tag{20}$$

$$t_{3cx} = t_0' + \frac{R_1 + R_2 + R_{3x}}{R_0}(t_c - t_0');\ R_0 = \frac{l_1}{\lambda_1} + \frac{l_2 - l_1}{\lambda_2} + \frac{l_3 - l_2}{\lambda_3} + \frac{1}{\alpha};$$

$$R_x = x/\lambda_1;\ R_{2x} = (x - l_1)/\lambda_2;\ R_{3x} = (x - l_2)/\lambda_3.$$

$$\begin{aligned}
\Delta_0 = K_{1+}^2 K_{2+}^2\ \{&(1 + \mathrm{Bi}^{*2})\,\mathrm{ch}\,2\delta_+ + (1 - \mathrm{Bi}^{*2})\cos 2\delta_+ + \\
&+ \sqrt{2}\,\mathrm{Bi}^{*2}(\mathrm{sh}\,2\delta_+ + \sin 2\delta_+) + h_1^2[(1 + \mathrm{Bi}^{*2})\,\mathrm{ch}\,2\delta_{-1} + (1 - \mathrm{Bi}^{*2})\cos 2\delta_{-1} + \\
&+ \sqrt{2}\,\mathrm{Bi}^*(\mathrm{sh}\,2\delta_{-1} + \sin 2\delta_{-1})] + h_1^2 h_2^2[(1 + \mathrm{Bi}^{*2})\,\mathrm{ch}\,2\delta_{-2} + \\
&+ (1 - \mathrm{Bi}^{*2})\cos 2\delta_{-2} + \sqrt{2}\,\mathrm{Bi}^*(\mathrm{sh}\,2\delta_{-2} + \sin 2\delta_{-2})] + \\
&+ h_2^2[(1 + \mathrm{Bi}^{*2})\,\mathrm{ch}\,2\delta_{-3} + (1 - \mathrm{Bi}^{*2})\cos 2\delta_{-2} + \\
&+ \sqrt{2}\,\mathrm{Bi}^*(\mathrm{sh}\,2\delta_{-3} + \sin 2\delta_{-3})] + 2h_1h_2[(1 + \mathrm{Bi}^{*2})\,\mathrm{ch}\,2(\delta_3 + \delta_1)\cos 2\delta_2 + \\
&+ (1 - \mathrm{Bi}^{*2})\,\mathrm{ch}\,2\delta_2\cos(\delta_3 + \delta_1) + \sqrt{2}\,\mathrm{Bi}^*(\mathrm{sh}\,2(\delta_3 + \delta_1)\cos 2\delta_2 + \\
&+ \mathrm{ch}\,2\delta_2\sin 2(\delta_3 + \delta_1))] + 2h_1h_2[(1 + \mathrm{Bi}^{*2})\,\mathrm{ch}\,2(\delta_3 - \delta_1)\cos 2\delta_2 + \\
&+ (1 - \mathrm{Bi}^{*2})\,\mathrm{ch}\,2\delta_2\cos(\delta_3 - \delta_1) + \sqrt{2}\,\mathrm{Bi}^* \times \\
&\times (\mathrm{sh}\,2(\delta_3 - \delta_1)\cos 2\delta_2 + \mathrm{ch}\,2\delta_2\sin 2(\delta_3 - \delta_1))] + \\
&+ 2h_1[(1 + \mathrm{Bi}^{*2})\,\mathrm{ch}\,2(\delta_3 + \delta_2)\cos 2\delta_1 + (1 - \mathrm{Bi}^{*2})\,\mathrm{ch}\,2\delta_1\cos 2(\delta_3 + \delta_2) + \\
&+ \sqrt{2}\,\mathrm{Bi}^*(\mathrm{sh}\,2(\delta_3 + \delta_2)\cos 2\delta_1 + \mathrm{ch}\,2\delta_1\sin 2(\delta_3 + \delta_2))] + \\
&+ 2h_2[(1 + \mathrm{Bi}^{*2})\,\mathrm{ch}\,2\delta_3\cos 2(\delta_2 + \delta_1) + (1 - \mathrm{Bi}^{*2})\,\mathrm{ch}\,2(\delta_2 + \delta_1)\cos 2\delta_3 + \\
&+ \sqrt{2}\,\mathrm{Bi}^*(\mathrm{ch}\,2(\delta_3 + \delta_1)\sin 2\delta_3 + \mathrm{sh}\,2\delta_3\cos 2(\delta_2 + \delta_1))] + \\
&+ 2h_1^2h_2[(1 + \mathrm{Bi}^{*2})\,\mathrm{ch}\,2\delta_3\cos 2(\delta_2 - \delta_1) + (1 - \mathrm{Bi}^{*2})\,\mathrm{ch}\,2(\delta_2 - \delta_1)\cos 2\delta_3 + \\
&+ \sqrt{2}\,\mathrm{Bi}^*(\mathrm{sh}\,2\delta_3\cos 2(\delta_2 - \delta_1) + \mathrm{ch}\,2(\delta_2 - \delta_1)\sin 2\delta_3)] + \\
&+ 2h_2^2h_1[(1 + \mathrm{Bi}^{*2})\,\mathrm{ch}\,2(\delta_3 - \delta_2)\cos 2\delta_1 + (1 - \mathrm{Bi}^{*2})\,\mathrm{ch}\,2\delta_1\cos 2(\delta_3 - \delta_2) + \\
&+ \sqrt{2}\,\mathrm{Bi}^*(\mathrm{sh}\,2(\delta_3 - \delta_2)\cos 2\delta_1 + \mathrm{ch}\,2\delta_1\sin 2(\delta_3 - \delta_2))]\};
\end{aligned} \tag{21}$$

$$\delta_1 = \sqrt{\omega/2a_1}\,l_1;\ \delta_2 = \sqrt{\omega/2a_2}\,(l_2 - l_1);\ \delta_3 = \sqrt{\omega/2a_3}\,(l_3 - l_2);$$

$$h_1 = K_{1-}/K_{1+};\ h_2 = K_{2-}/K_{2+}.$$

In the above Δ_x is obtained from Δ_0 by replacing δ_1 with $\delta_{1x} = \sqrt{\omega/2a_1}\,(l_1 - x)$; Δ_{2x} is obtained from Δ_0 if in Δ_x we put $K_{1-} = 0$, $K_{1+} = 1$, $\delta_1 = 0$, and replace δ_2 with $\delta_{2x} = \sqrt{\omega/2a_2}\,(l_2 - x)$; Δ_{3x} is obtained from Δ_0 if we put $K_{1-} = 0$, $K_{2-} = 0$, $K_{1+} = 1$, $K_{2+} = 1$, $\delta_2 = 0$, $\delta_1 = 0$, and replace δ_3 with $\sigma_{3x} = \sqrt{x/2a_3}\,(l_3 - x)$;

$$\delta_+ = \delta_1 + \delta_2 + \delta_3;\ \delta_{-1} = \delta_2 + \delta_3 - \delta_1;\ \delta_{-2} = \delta_1 + \delta_3 - \delta_2;$$
$$\delta_{-3} = \delta_1 + \delta_2 - \delta_3;$$
$$K_{1+} = K_1 + 1;\ K_{1-} = K_1 - 1;\ K_{2+} = K_2 + 1;\ K_{2-} = K_2 - 1;$$
$$K_1 = \sqrt{\lambda_1 c_1 \gamma_1/\lambda_2 c_2 \gamma_2};\ K_2 = \sqrt{\lambda_2 c_2 \gamma_2/\lambda_3 c_3 \gamma_3};$$
$$\beta_{1x} = \arccos \frac{A_x}{\sqrt{\Delta_x}};\ \beta_2 = \arccos \frac{A_{2x}}{\sqrt{\Delta_{2x}}};$$
$$\beta_3 = \arccos \frac{A_{3x}}{\sqrt{\Delta_{x3}}};\ \gamma = \beta_{1x}|_{x=0};$$
$$A_x = \operatorname{ch}\delta_x^+ \cos\delta_x^+ - \operatorname{sh}\delta_x^+ \sin\delta_x^+ + h_1(\operatorname{ch}\delta_x^{-1}\cos\delta_x^{-1} - \operatorname{sh}\delta_x^{-1}\sin\delta_x^{-1}) +$$
$$+ h_2 h_1(\operatorname{ch}\delta_x^{-2}\cos\delta_x^{-2} - \operatorname{sh}\delta_x^{-2}\sin\delta_x^{-2}) + h_2(\operatorname{ch}\delta_x^{-3}\cos\delta_x^{-3} - \operatorname{sh}\delta_x^{-3}\sin\delta_x^{-3}) +$$
$$+ \sqrt{2}\,\mathrm{Bi}^*(\operatorname{sh}\delta_x^+ \cos\delta_x^+ + h_1 \operatorname{sh}\delta_x^{-1}\cos\delta_x^{-1} +$$
$$+ h_1 h_2 \operatorname{sh}\delta_x^{-2}\cos\delta_x^{-2} + h_2 \operatorname{sh}\delta_x^{-3}\cos\delta_x^{-3});$$
$$A_{2x} = \operatorname{ch}\delta_+^x \cos\delta_+^x - \operatorname{sh}\delta_+^x \sin\delta_+^x + h_2(\operatorname{ch}\delta_-^x\cos\delta_-^x - \operatorname{sh}\delta_-^x \sin\delta_-^x) +$$
$$+ \sqrt{2}\,\mathrm{Bi}^*(\operatorname{sh}\delta_+^x\cos\delta_+^x + h_2 \operatorname{sh}\delta_-^x\cos\delta_-^x);$$
$$\delta_+^x = \delta_3 + \delta_{2x};\ \delta_-^x = \delta_3 - \delta_{2x};$$
$$A_{3x} = \operatorname{ch}\delta_{3x}\cos\delta_{3x} - \operatorname{sh}\delta_{3x}\sin\delta_{3x} + \sqrt{2}\,\mathrm{Bi}^* \operatorname{sh}\delta_{3x}\cos\delta_{3x};$$
$$\Delta_x' = K_{1+}^2 K_{2+}^2\ \{[(1+\mathrm{Bi}^{*2})\operatorname{ch}2\delta_+ - (1-\mathrm{Bi}^{*2})\cos 2\delta_+ +$$
$$+ \sqrt{2}\,\mathrm{Bi}^*(\operatorname{sh}2\delta_+ - \sin 2\delta_+)] + [(1+\mathrm{Bi}^{*2})\operatorname{ch}2\delta_1 - (1-\mathrm{Bi}^{*2})\cos 2\delta_{-1} +$$
$$+ \sqrt{2}\,\mathrm{Bi}^*(\operatorname{sh}2\delta_{-1} - \sin 2\delta_{-1})]\,h_1^2 +$$
$$+ [(1+\mathrm{Bi}^{*2})\operatorname{ch}2\delta_{-2} - (1-\mathrm{Bi}^{*2})\cos 2\delta_{-2} +$$
$$+ \sqrt{2}\,\mathrm{Bi}^*(\operatorname{sh}2\delta_{-2} - \sin 2\delta_{-2})]\,h_1^2 h_2^2 +$$
$$+ [(1+\mathrm{Bi}^{*2})\operatorname{ch}2\delta_{-3} - (1-\mathrm{Bi}^{*2})\cos 2\delta_{-3} + \sqrt{2}\,\mathrm{Bi}^*(\operatorname{sh}2\delta_{-3} - \sin 2\delta_{-3})]\,h_2^2 +$$
$$+ [(1+\mathrm{Bi}^{*2})\operatorname{ch}2(\delta_3+\delta_{1x})\cos 2\delta_2 - (1-\mathrm{Bi}^{*2})\operatorname{ch}2\delta_2\cos 2(\delta_3+\delta_{1x}) +$$
$$+ \sqrt{2}\,\mathrm{Bi}^*(\operatorname{sh}2(\delta_3+\delta_{1x})\cos 2\delta_2 - \operatorname{ch}2\delta_2 \sin 2(\delta_3+\delta_{1x}))]\,2h_1h_2 +$$
$$+ [(1+\mathrm{Bi}^{*2})\operatorname{ch}2(\delta_3-\delta_{1x})\cos 2\delta_2 - (1-\mathrm{Bi}^{*2})\operatorname{ch}2\delta_2\cos 2(\delta_3-\delta_{1x}) +$$
$$+ \sqrt{2}\,\mathrm{Bi}^*(\operatorname{sh}2(\delta_3-\delta_{1x})\cos 2\delta_2 - \operatorname{ch}2\delta_2 \sin 2(\delta_3-\delta_{1x}))]\,2h_1h_2 +$$
$$+ [(1+\mathrm{Bi}^{*2})\operatorname{ch}2(\delta_2+\delta_3)\cos 2\delta_{1x} - (1-\mathrm{Bi}^{*2})\operatorname{ch}2\delta_{1x}\cos 2(\delta_2+\delta_3) +$$
$$+ \sqrt{2}\,\mathrm{Bi}^*(\operatorname{ch}2\delta_{1x}\sin 2(\delta_3+\delta_2) - \operatorname{sh}2(\delta_3+\delta_2)\cos 2\delta_{1x})]\,2h_1 +$$
$$+ [(1+\mathrm{Bi}^{*2})\operatorname{ch}2\delta_2\cos 2(\delta_3+\delta_{1x}) - (1-\mathrm{Bi}^{*2})\operatorname{ch}2(\delta_3+\delta_{1x})\cos 2\delta_2 +$$
$$+ \sqrt{2}\,\mathrm{Bi}^*(\operatorname{ch}2(\delta_{1x}+\delta_2)\sin 2\delta_3 - \operatorname{sh}2\delta_3\cos 2(\delta_{1x}+\delta_2))]\,2h_2 +$$
$$+ [(1+\mathrm{Bi}^{*2})\operatorname{ch}2\delta_3\cos 2(\delta_2-\delta_{1x}) - (1-\mathrm{Bi}^{*2})\operatorname{ch}2(\delta_2-\delta_{1x})\cos 2\delta_3 +$$
$$+ \sqrt{2}\,\mathrm{Bi}^*(\operatorname{ch}2(\delta_2-\delta_{1x})\sin 2\delta_3 - \operatorname{sh}2\delta_3\cos 2(\delta_2-\delta_{1x}))]\,2h_1^2h_2 +$$
$$+ [(1+\mathrm{Bi}^{*2})\operatorname{ch}2\delta_{1x}\cos 2(\delta_3-\delta_2) - (1-\mathrm{Bi}^*)\operatorname{ch}2(\delta_3-\delta_2)\cos 2\delta_{1x} +$$
$$+ \sqrt{2}\,\mathrm{Bi}^*(\operatorname{ch}2\delta_{1x}\sin 2(\delta_3-\delta_2) - \operatorname{sh}2(\delta_3-\delta_2)\cos 2\delta_{1x})]\,2h_1h_2^2\}. \qquad (22)$$

In this case Δ_{2x}' is obtained from Δ_x' by putting $K_{1-} = 0$, $\delta_1 = 0$, $K_{1+} = 1$, and replacing δ_2 with $\delta_{2x} = \sqrt{\omega/2a_2}\,(l_2 - x)$; Δ_{3x} is obtained from Δ_x' by putting $K_{1-} = 0$, $K_{2-} = 0$, $K_{1+} = 1$, $K_{2-} = 1$, $\delta_1 = 0$, $\delta_2 = 0$, and replacing δ_3 with $\delta_{3x} = \sqrt{\omega/2a_3}\,(l_3 - x)$.

From solutions (15), (16), and (17) we obtain the solution for a two-layer slab, if we put $K_{1-} = 0$, $K_{1+} = 2$, $\delta_1 + \delta_2 = \delta_1$. The solution has the form:

$$t_1(x,\ \tau) = t_{1cx} + t_m A(x,\ \omega)\cos(\omega\tau - \gamma + \beta_x); \quad (23)$$

$$t_2(x,\ \tau) = t_{2xc} + t_m A_2(x,\ \omega)\cos(\omega\tau - \gamma + \beta_{2x}). \quad (24)$$

For the heat fluxes we obtain the expressions:

$$q_1(x,\ \tau) = Q_{1c} + \sqrt{\lambda_1 c_1 \gamma_1 \omega}\ t_m A(x,\ \omega)\cos(\omega\tau - \gamma + \beta_x + \beta_x')\sqrt{\Delta_x'/\Delta_x}\ ; \quad (25)$$

$$q_2(x,\ \tau) = Q_{2c} + \sqrt{\lambda_2 c_2 \gamma_2 \omega}\ t_m A_2(x,\ \omega)\cos(\omega\tau - \gamma + \beta_{2x} + \beta_{2x}')\sqrt{\Delta_{2x}'/\Delta_{2x}}\ ; \quad (26)$$

$$A_1(x,\ \omega) = \sqrt{\frac{\Delta_x}{\Delta_0}};\ A_2(x,\ \omega) = \frac{2K}{K+1}\left(\frac{\Delta_{2x}}{\Delta_0}\right)^{1/2};$$

$$\Delta_0 = [(1+\mathrm{Bi}^{*2})\,\mathrm{ch}\,2\delta_+ + (1-\mathrm{Bi}^{*2})\cos 2\delta_+ +$$
$$+\sqrt{2}\,\mathrm{Bi}^*(\mathrm{sh}\,2\delta_+ + \sin 2\delta_+)] + h^2[(1+\mathrm{Bi}^{*2})\,\mathrm{ch}\,2\delta_- + (1-\mathrm{Bi}^{*2})\cos 2\delta_- +$$
$$+\sqrt{2}\,\mathrm{Bi}^*(\mathrm{sh}\,2\delta_- + \sin 2\delta_-)] + 2h[(1+\mathrm{Bi}^{*2})\,\mathrm{ch}\,2\delta_2\cos 2\delta_1 +$$
$$+(1-\mathrm{Bi}^{*2})\,\mathrm{ch}\,2\delta_1\cos 2\delta_2 + \sqrt{2}\,\mathrm{Bi}^*(\mathrm{sh}\,2\delta_2\cos 2\delta_1 + \mathrm{ch}\,2\delta_1\sin 2\delta_2)]; \quad (27)$$

$$t_{1cx} = t_0' + \frac{R_{1x}}{R_0}(t_c - t_0');\ t_{2cx} = t_0' + \frac{R_1 + R_{2x}}{R_0}(t_c - t_0');$$

$$\Delta_x' = [(1+\mathrm{Bi}^{*2})\,\mathrm{ch}\,2\delta_+ - (1-\mathrm{Bi}^{*2})\cos 2\delta_+ + \sqrt{2}\,\mathrm{Bi}^*(\mathrm{sh}\,2\delta_+ - \sin 2\delta_+)] +$$
$$+h^2[(1+\mathrm{Bi}^{*2})\,\mathrm{ch}\,2\delta_- - (1-\mathrm{Bi}^{*2})\cos 2\delta_- + \sqrt{2}\,\mathrm{Bi}^*(\mathrm{sh}\,2\delta_- - \sin 2\delta_-)] -$$
$$-2h[(1+\mathrm{Bi}^{*2})\,\mathrm{ch}\,2\delta_2\cos 2\delta_{1x} - (1-\mathrm{Bi}^{*2})\,\mathrm{ch}\,2\delta_{1x}\cos 2\delta_2 + \sqrt{2}\,\mathrm{Bi}^*(\mathrm{sh}\,2\delta_2 \times$$
$$\times\cos 2\delta_{1x} - \mathrm{ch}\,2\delta_{1x}\sin 2\delta_2)];$$

$$\delta_{1x} = \sqrt{\omega/2a_1}\,(l_1 - x);\ \delta_+ = \delta_{1x} + \delta_2;\ \delta_- = \delta_2 - \delta_{1x};$$

Δ_x is obtained from Δ_0 by replacing δ_1 with δ_{1x};

$$\Delta_{2x} = (1+\mathrm{Bi}^{*2})\,\mathrm{ch}\,2\delta_{2x} + (1-\mathrm{Bi}^{*2})\cos 2\delta_{2x} + \sqrt{2}\,\mathrm{Bi}^*(\mathrm{sh}\,2\delta_{2x} + \sin 2\delta_{2x});$$
$$\Delta_{2x}' = (1+\mathrm{Bi}^{*2})\,\mathrm{ch}\,2\delta_{2x} - (1-\mathrm{Bi}^{*2})\cos 2\delta_{2x} + \sqrt{2}\,\mathrm{Bi}^*(\mathrm{sh}\,2\delta_{2x} - \sin 2\delta_{2x});$$
$$K_\varepsilon = \sqrt{\lambda_1 c_1\gamma_1/\lambda_2 c_2\gamma_2};\ h = (K_\varepsilon - 1)/K_\varepsilon + 1);\ \delta_{2x} = \sqrt{\omega/2a_2}\,(l_2 - x);\ \mathrm{Bi}^* =$$
$$= \alpha/\sqrt{\lambda c\gamma\omega}\,.$$

Analysis of solutions. The reference value for calculating the heat transmission through an outside wall, according to Soviet Construction Norms and Specifications (CNS) [9], is $D = \sum_{i=1}^{n} R_i s_i$ (thermal inertia). R_i is the thermal resistance of the i-th layer, and s_i is the coefficient of assimilation of heat. We shall use solutions (9) and (12) to determine s for a two-layer wall (second layer infinitely thick):

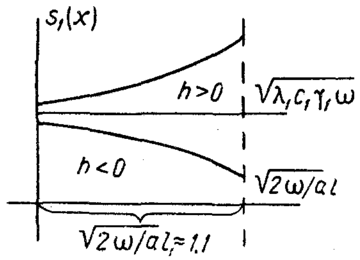
$$s(x) = \frac{q(x)_{\max}}{t(x)_{\max}};$$

$$s_1(x) = \sqrt{\lambda_1 c_1 \gamma_1 \omega}\,\big(\{1 + 2h\exp(-2l_1 + 2x)\sqrt{\omega/2a_1}\cos 2(l_1 - x)\sqrt{\omega/2a_1} +$$
$$+ h^2\exp[(-4l_1 + 4x)\sqrt{\omega/2a_1}]\}\{1 - 2h\exp[(-2l_1 + 2x)\times$$
$$\times\sqrt{\omega/2a_1}]\cos 2(l_1 - x)\sqrt{\omega/2a_1} + h^2\exp[(-4l_1 + 4x)\sqrt{\omega/2a_1}]\}^{-1}\big)^{1/2};$$
$$s_2(x) = \sqrt{\lambda_2 c_2 \gamma_2 \omega}\,.$$

It follows that, for finite layers (even for an infinite thickness of construction), the value of s does not coincide with that recommended by CNS [9]. Let us consider (1) in more detail. In [9] the recommended value for s when $D \geq 1$ is $s = \sqrt{\lambda c\gamma\omega}$. But $2l_1\sqrt{\omega/2a_1} = \sqrt{2}\ (l_1/\lambda_1)\sqrt{\lambda c\gamma\omega} = \sqrt{2}\,D$. For any x, expression (1) differs from $\sqrt{\lambda c\gamma\omega}$, and coincides with it for $2(l_1 - x)\sqrt{\omega/2a_1} = (2K+1)\pi/2$. When $2l_1\sqrt{\omega/2a_1} = \pi/2$, $D \cong 1.1$, and consequently for this kind of layer [9] recommends $s = \sqrt{\lambda_1 c_1\gamma_1\omega}$. At the boundary between the layers we have, from (1):

$$s_1(l_1) = \sqrt{\lambda_1 c_1 \gamma_1 \omega}\,\frac{1-h}{1+h} = \sqrt{\lambda_1 c_1 \gamma_1 \omega}\,\frac{1}{K_\varepsilon}\,.$$

Thus, $s_1(l_1)$ does not depend on the value of l_1, and at the boundary between the layers, for any thickness, if the materials are the same, will be a constant determined by the material properties. If we represent (1) graphically, we obtain the curves shown in the figure. Using solutions (23), (24), (25), and (26) to determine the value of s, we get:



Variation of s = q_{max}/t_{max} in a layer with D ≈ 1.

$$s_1(x) = \sqrt{\lambda_1 c_1 \gamma_1 \omega}\sqrt{\Delta'_x/\Delta_x}, \tag{28}$$

$$s_2(x) = \sqrt{\lambda_2 c_2 \gamma_2 \omega}\sqrt{\Delta'_{2x}\Delta_{2x}}. \tag{29}$$

Let the second layer be of such a thickness that (D > 1) $2\sqrt{\omega/2a_1}\,(l_2 - l_1) > \pi/2$. This means that, using [9], we should take $s = \sqrt{\lambda_2 c_2 \gamma_2 \omega}$. At the inside surface $s = q_{max}/t_{max} = \alpha$, and it increases or decreases up to the boundary between the layers (if D ≅ 1.1). Thus, the $\sqrt{\lambda c \gamma \omega}$ does not characterize the relation q_{max}/t_{max} at any point in the layer, but if we take the integral mean value of s for a layer with D > 1.1, it may be several times greater or less than the value of $\sqrt{\lambda c \gamma \omega}$.

We shall find the integral mean $\bar{s}(x)$ for three different materials in a layer with $2\delta_+ = \sqrt{2}\,D = \pi/2$, i.e., D ≈ 1.1:

a) a natural stone wall, $\sqrt{\lambda c \gamma \omega} = 20.6$, x = 0.11, Bi* = 0.36, $s(x)/\sqrt{\lambda c \gamma \omega} \cong 0.4$;

b) a brick wall, $\sqrt{\lambda c \gamma \omega} = 7.5$, x = 0.7, Bi* = 1, $\bar{s}(x)/\sqrt{\lambda c \gamma \omega} \approx 1$;

c) foam concrete, $\gamma = 400$, $\sqrt{\lambda c \gamma \omega} = 1.58$, x = 0.6, Bi* = 4.74, $s(x)/\sqrt{\lambda c \gamma \omega} \cong 2.3$.

It follows from formulas (28) and (29) that $s = \sqrt{\lambda c \gamma \omega}$ will be close to q_{max}/t_{max} for materials for which $\sqrt{\lambda c \gamma \omega}$ is close to α.

The thermal inertia characteristic D is introduced to allow for unsteady nature of heat transfer. The outside design temperature is chosen according to the value of D (average for the coldest five-day period, or average for the coldest 24 hours). For the same average temperature, however, the amplitudes may differ (temperature varies from −3°C to 3°C, and from −10°C to 10°C during 24 hours, and in both cases the average temperature is zero).

It follows from (23) to (26) that the temperature and heat flux oscillations at the inside surface (the half-period value of the flux is taken) depend on t_m (amplitude of oscillations of outside air), $2\delta_+ = \sqrt{2}\,D$ (thermal inertia), $2\delta_i = \sqrt{2}\,D_i$ (thermal inertia of layer), and also on the order of the layers and the properties of the materials used.

As an example, we shall consider the variable components of temperature and heat flux at the inside surface of a two-layer wall.

From (24) and (26) we obtain

$$t_2(l_2, \tau) = 2\sqrt{2}\,\frac{K}{(K+1)\sqrt{\Delta_0}}\,t_m \cos(\omega\tau - \gamma), \tag{30}$$

$$q_2(l_2, \tau) = 2\sqrt{2}\,\frac{K\alpha}{(K+1)\sqrt{\Delta_0}}\,t_m \cos(\omega\tau - \psi). \tag{31}$$

Hence it follows that, for a given arrangement of materials (K does not change) and a given amplitude of the oscillations of the outside air (t_m = const), the amplitude of the oscillations at the inside surface is determined by $\sqrt{\Delta_0}$ (α is considered constant).

$$\begin{aligned}\Delta_0 = {} & [(1+\mathrm{Bi}^{*2})\,\mathrm{ch}\,2\delta_+ + (1-\mathrm{Bi}^{*2})\cos 2\delta_+ + \sqrt{2}\,\mathrm{Bi}^*(\mathrm{sh}\,2\delta_+ + \sin 2\delta_+)] + \\ & + 2h[(1+\mathrm{Bi}^{*2})\,\mathrm{ch}\,2\delta_2 \cos 2\delta_1 + (1-\mathrm{Bi}^{*2})\,\mathrm{ch}\,2\delta_1 \cos 2\delta_2 - \sqrt{2}\,\mathrm{Bi}^*\,(\mathrm{sh}\,2\delta_2 \times \\ & \times \cos 2\delta_1 + \mathrm{ch}\,2\delta_1 \sin 2\delta_2)] + h^2[(1+\mathrm{Bi}^{*2})\,\mathrm{ch}\,2\delta_- + (1-\mathrm{Bi}^2)\cos 2\delta_- + \sqrt{2}\,\mathrm{Bi}^* \times \\ & \times (\mathrm{sh}\,2\delta_- + \sin 2\delta_-)].\end{aligned}$$

The first bracket depends on $2\delta_+ = \sqrt{2}\,D$ (thermal inertia of wall). The bracket preceded by 2h and h^2 on $2\delta_1 = \sqrt{2}\,D_1$ and $2\delta_2 = \sqrt{2}\,D_2$ and on the relation between them.

If we vary the thicknesses of the layers, keeping the $2\delta_+$ of the wall constant, then the first bracket does not change, but the other two do. Therefore, the amplitude at the inside surface and the heat flux also change.

Hence for walls with D < a, Δ_0 may be larger than for walls with D > a. And if a is a value that divides walls into classes in terms of massiveness, then, for the same R, the better of two walls in a thermal sense might be judged unfavorable.

Let us examine three two-layer walls a, b, and c (Table 1).

TABLE 1

Parameters for determining losses in walls

Parameter	First layer			Second layer		
	a	*b*	*c*	*a*	*b*	*c*
s	12.50	6.45	3.02	5.00	1.30	6.45
λ	1.25	0.50	0.22	0.40	0.11	0.50
R	0.03	0.572	0.10	0.715	0.20	0.592
D	0.375	3.69	0.302	3.575	0.26	3.8184
γ	2400	1000	800	1200	300	1600

Following CNS, we shall determine the suitability of these walls for the following conditions.

Average temperature during coldest 24 hours −23°C; average temperature during coldest 5-day period −15°C. For light structures R should not be less than $R_{tr} = (t_1 - t_0)nb/\alpha\Delta t_0$. For wall a, D = 3.95, R = 0.878, R_{tr} = 0.911; for wall b, D = 3.95, R = 0.905, R_{tr} = 1.002; for wall c, D = 4.12, R = 0.825, R_{tr} = 0.822. Therefore, according to CNS, only wall c is satisfactory for the given locality.

Let us find the ratio of amplitude of the oscillations at the inside surface to that at the outside surface, using (30) and (31). For a it is 0.058, for b − 0.022, and for c − 0.05.

Thus, the construction with the greatest thermal resistance and the greatest inertia to variable thermal effects is judged unsuitable, while the worst is recommended. This leads in some cases to an unnecessary over-or underestimating of the cost of materials and the thickness of the wall. For materials with markedly different characteristics this discrepancy will be even greater. We present a table of three-layer and two-layer walls calculated in accordance with (14)-(19).

The following quantities are given in Table 2: the thermal resistance $R_{total} = l_1/\lambda_1 + (l_2 - l_1)/\lambda_2 + (l_3 - l_2)/\lambda_3 + 1/\alpha$; $D_{total} = 2\delta_+ = \sqrt{2}\, D$ (D is the thermal inertia according to CNS); t_{max}/t_m − the ratio of the amplitudes at the inside and outside surfaces; Q_{unst}/Q_{st} − the ratio of the unsteady to the steady components of the heat flux. In this case each line gives two results: the first is the ratio of the variable component during a half-period to the constant component of the flux through a brick wall for the same design temperatures; and the second is the ratio of the variable component of the flux during a half-period to the steady component of the flux through the same wall. The ratios are given as percentages, the factor $t_m/(t_i - t_0)$ is taken out and placed at the head of the table, which also gives the ratio of the sum of the terms in (21), which take into account the order of the layers and the ratio of their $D_i = \sqrt{\omega/2a_i}\,(l_i - l_{i-1})$, to the term determined by $D = \sum_{i=1}^{n} R_i s_i$.

These calculations (last column of Table 2) well illustrate the fact that the quantity D, which in Δ_0 determines the value of the terms not containing h, is not characteristic of the thermal inertia (massiveness of wall), since the numerical value of D does not determine the value of the heat flux and the amplitude at the inside surface in the presence of variable thermal effects.

The oscillations of the temperature and heat flux under changing thermal conditions are 1.5 times less for wall 2 than for wall 1. Therefore the latter has greater thermal inertia, although it has a smaller D than the former.

It follows from the foregoing that the heat transfer in outside walls must be calculated on the basis of more accurate formulas than those recommended in CNS. The calculations may be done directly from (14)-(19) or (23)-(26). The functions entering into the formulas have been tabulated, and nomograms may also be constructed.

Method of constructing nomograms. The first four brackets in (21) and the first two in (27) have different indices, so that one nomogram is required for these six terms. Another nomogram is required for the last six brackets in (21) and the last bracket in (27).

Having established the range of variation of the quantities occurring in (27) and (21) $\delta_1 = \sqrt{\omega/2a_1}\, l_1$, $\delta_2 = \sqrt{\omega/2a_2}\,(l_2 - l_1)$, $\delta_+ = \delta_1 + \delta_2$, $\delta_- = \delta_2 - \delta_1$, $h = (K - 1)/(K + 1)$, $Bi^* = \alpha/\sqrt{\lambda_2 c_2 \gamma_2 \omega}$, let us construct in the coordinate system Bi^*, δ the family of lines γ = const, where

$$\gamma = Bi^{*2}(\text{ch } 2\delta - \cos 2\delta) + Bi^*(\text{sh } 2\delta + \sin 2\delta)\sqrt{2} + (\text{ch } 2\delta + \cos 2\delta).$$

TABLE 2

Calculation of walls in accordance with formulas (14) to (19)

Wall	Material and thickness of layer			R_{total}	D_{total}	$\frac{t_{max}}{t_m}$	$\frac{Q_{unst}}{Q_{st}}$ $\frac{t_i - t_o}{t_m}$	$\frac{\Sigma AD}{D}$ 100%
	1st	2nd	3rd					
1	Lime plaster 20 mm	Brick 510 mm	Plaster on inside face 20 mm	0.922	6.20	0.015	6.6 6.6	3
2	Lime plaster 20 mm	Fibrolite 260 mm	Plaster on inside face 20 mm	2.793	5.60	0.0098	4.31 13	89
3	Reinforced concrete 40 mm	Foam concrete 200 mm	Reinforced concrete 20 mm	2.321	3.60	0.029	12.7 32	89
4	Reinforced concrete 25 mm	Mineral wool 150 mm	Reinforced concrete 20 mm	2.949	2.899	0.033	14.5 48.6	61
5	Lime plaster 20 mm	Slag concrete 380 mm		1.004	4.54	0.038	16.7 18.2	35
6	Lime plaster 20 mm	Expanded-clay concrete 400 mm		0.960	5.206	0.024	11.8 12.9	17
7	Lime plaster 20 mm	Lightweight lime concrete 280 mm		0.960	3.70	0.056	24.6 25.6	75

We find the γ corresponding to the Bi* and δ calculated for the construction. To calculate the first four brackets of (21), four readings must be taken at the same Bi*. The sum of the four brackets for a three-layer wall is:

$$A(\gamma) = \gamma_1 + h_1^2 \gamma_2 + h_1^2 h_2^2 \gamma_3 + h_2^2 \gamma_4,$$

and for a two-layer wall (27):

$$A(\gamma) = \gamma_1 + h^2 \gamma_2.$$

For the last six brackets of (21), three families of lines are constructed in the coordinate system δ_1, δ_2:

$$\beta = \operatorname{ch} \delta_1 \cos \delta_2 - \operatorname{ch} \delta_2 \cos \delta_1 = \text{const},$$

$$\alpha = \sqrt{2}\,(\operatorname{sh} \delta_1 \cos \delta_2 + \operatorname{ch} \delta_2 \sin \delta_1) = \text{const},$$

$$\omega = \operatorname{ch} \delta_1 \cos \delta_2 + \operatorname{ch} \delta_2 \cos \delta_1 = \text{const}.$$

For convenience each family can be represented in a separate plane.

The numerical value of the bracket $B = \omega + \mathrm{Bi}^*\alpha + \mathrm{Bi}^{*2}\beta$. The remaining brackets are calculated in a similar way.

Finally, it should be noted that the quantity $D = \sum_{i=1}^{n} R_i s_i$, assumed to be characteristic of the thermal inertia, is actually not. Therefore choosing design temperatures (to determine the thermal resistance) on the basis of D is incorrect. The existing method of choosing design temperatures leads either to wastage of material or to unsatisfactory construction. The proposed method of calculation permits the selection of materials and thicknesses in accordance with the known requirements made on the wall, and the determination of the nonuniformity of heat losses under variable thermal conditions, thus allowing a more rational choice of the power and refrigeration capacity of the air-conditioning plant.

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